Completeness of Automata with respect to Equivalence Relations

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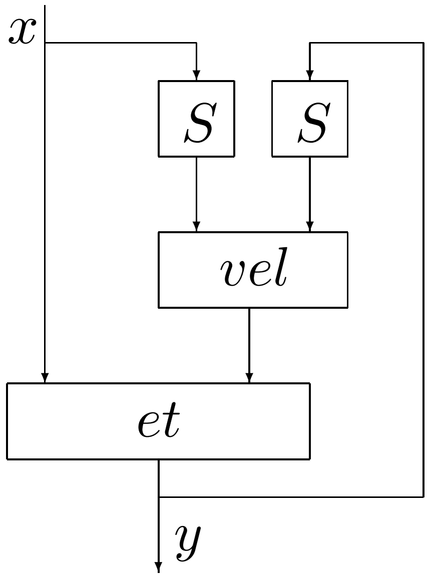
Logical Nets

logical nets induce functions $F : (\{0,1\}^n)^* \rightarrow (\{0,1\}^m)^*$



input	output
0	0
1	0
00	00
01	00
10	00
11	01

input	output
000	000
001	000
010	000
011	001
100	000
101	000
110	010
111	011

input	output
...	...
11011	01001
11100	01100
11101	01100
11111	01111
...	...

Automata – Definition

Definition:

A (deterministic finite) <u>automaton</u> is a quintuple $\mathcal{A} = (X, Y, Z, z_0, \delta, \gamma)$ where

– X, Y and Z are finite non-empty sets
of input and output symbols and states, respectively,

– $z_0 \in Z$ is the initial state,

– $\delta : Z \times X \to Z$ and $\gamma : Z \times X \to Y$ are functions
called the transition and output function.

Extension to $\delta^* : Z \times X^* \to Z$ and $\gamma^* : Z \times X^* \to Y^*$ by

– $\delta^*(z, \lambda) = z$ and $\delta^*(z, wa) = \delta(\delta^*(z, w), a)$

– $\gamma^*(z, \lambda) = \lambda$ and $\gamma^*(z, wa) = \gamma^*(z, w)\gamma(\delta^*(z, w), a)$

for $w \in X^*$ and $a \in X$

Automata – Example

$$\mathcal{A} = (\{0,1\}, \{0,1\}, \{00,10,11\}, 00, \delta, \gamma)$$

δ	0	1
00	00	10
10	00	11
11	00	11

γ	0	1
00	0	0
10	0	1
11	0	1

Sequential Functions – Definition

Definition: A function $F : X^* \to Y^*$ is called sequential, if the following conditions are satisfied:

- $|F(p)| = |p|$ for any $p \in X^*$,
- for any word $p \in X^*$, there is a function F_p such that $F(pq) = F(p)F_p(q)$ for any $q \in X^*$,
- the set $\{F_p \mid p \in X^*\}$ is finite.

$F(\lambda) = \lambda$,
$F(x_1x_2\ldots x_n) = y_1y_2\ldots y_n$ with $x_i \in X$ and $y_i \in Y$ for $1 \leq i \leq n$

Sequential Functions – Example

$F : \{0,1\}^* \to \{0,1\}^*$

$$F(x_1x_2\dots x_n) = y_1y_2\dots y_n \text{ with } y_i = \begin{cases} 1 & i \geq 2,\ x_{i-1} = x_i = 1, \\ 0 & \text{otherwise} \end{cases}$$

$$F'(x_1x_2\dots x_n) = y_1y_2\dots y_n \text{ with } y_i = \begin{cases} 1 & i = 1,\ x_1 = 1, \\ 1 & i \geq 2,\ x_{i-1} = x_i = 1, \\ 0 & \text{otherwise} \end{cases}$$

$F_p = F$ for $p \in \{0,1\}^*0$

$F_p = F'$ for $p \in \{0,1\}^*1$

Sequential Functions – Description

sequential function $F : (\{0,1\}^m)^* \to \{0,1\}^*$,

arity of F – $arity(F) = m$

$F(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n$
where $x_i \in \{0,1\}^m$ and $y_i \in \{0,1\}$ for $1 \le i \le n$

$x_i = (x_{i1}, x_{i2}, \dots, x_{im})$ for $1 \le i \le n$	$p_j = x_{1j} x_{2j} \dots x_{nj}$
$y_i = \varphi_F^i(x_1, x_2, \dots, x_i)$ $= \varphi_F^i(x_{11}, x_{12}, \dots, x_{1m}, x_{21}, \dots, x_{im})$	$F(p_1, p_2, \dots p_m)$

Operations on $\mathcal{F}$ I

$\mathcal{F}^m$ – set of all sequential functions $F : (\{0,1\}^m)^* \to \{0,1\}^*$ (of arity m)

$\mathcal{F} = \bigcup_{m \geq 0} \mathcal{F}^m$

For $F \in \mathcal{F}^m$ and $G \in \mathcal{F}^k$, we set

$$\begin{aligned}
(\zeta F)(p_1, p_2, \ldots, p_m) &= F(p_2, p_3, \ldots, p_m, p_1), \\
(\eta F)(p_1, p_2, \ldots, p_m) &= F(p_2, p_1, p_3, p_4, \ldots, p_m), \\
(\Delta F)(p_1, p_2, \ldots, p_{m-1}) &= F(p_1, p_1, p_2, \ldots, p_{m-1}), \\
(\nabla F)(p_1, p_2, \ldots, p_{m+1}) &= F(p_1, p_2, \ldots, p_m), \\
(F \circ G)(p_1, p_2, \ldots, p_{m+k-1}) &= F(G(p_1, p_2, \ldots p_k), p_{k+1}, \ldots, p_{k+m-1})
\end{aligned}$$

Operations on $\mathcal{F}$ II

$F \in \mathcal{F}^m$ depends delayed on its first variable if, for $i \geq 1$, φ_F^i does not depend on x_{i1}.

If $F \in \mathcal{F}^m$ depends delayed on its first variable, we define $\uparrow F$ by

$$
\begin{aligned}
&\varphi^1_{\uparrow F}(x_{12}, x_{13}, \ldots, x_{1m}) = \varphi^1_F(x_{11}, x_{12}, x_{13}, \ldots, x_{1m}),\\
&\varphi^t_{\uparrow F}(x_{12}, \ldots, x_{1m}, x_{22}, \ldots, x_{2m}, \ldots, x_{i2}, \ldots, x_{im})\\
&\qquad = \varphi^t_F(\varphi^1_{\uparrow F}(z_1), x_{12}, x_{13}, \ldots, x_{1m},\\
&\qquad\qquad \varphi^2_{\uparrow F}(z_2), x_{22}, x_{23}, \ldots, x_{2m},\\
&\qquad\qquad \ldots\\
&\qquad\qquad \varphi^{t-1}_{\uparrow F}(z_{t-1}), x_{t-1,2}, \ldots, x_{t-1,m},\\
&\qquad\qquad x_{t2}, x_{t3}, \ldots, x_{tm})
\end{aligned}
$$

Algebra $\mathcal{F}$

algebra $\underline{\mathcal{F}} = (\mathcal{F}, \{\zeta, \eta, \Delta, \nabla, \circ, \uparrow\})$

$[M]$ – subalgebra generated by $M \subseteq \mathcal{F}$ in $\underline{\mathcal{F}}$

algebra $\underline{\mathcal{F}'} = (\mathcal{F}, \{\zeta, \eta, \Delta, \nabla, \circ\})$

$< M >$ – subalgebra generated by $M \subseteq \mathcal{F}$ in $\underline{\mathcal{F}'}$

$P_k = \{f \mid f : \{0, 1, \ldots, k-1\}^m \to \{0, 1, \ldots, k-1\},\ m \geq 0\}$
algebra $\underline{P_k} = (P_k, \{\zeta, \eta, \Delta, \nabla, \circ\})$

Completeness with respect to an Equivalence Relation

Definition: Let ϱ be an equivalence relation on $\mathcal{F}$.
A subalgebra M of $\mathcal{F}$ is called a ϱ-algebra iff $M \cap K \neq \emptyset$ for all equivalence classes K of ϱ.

Definition: Let ϱ be an equivalence relation on $\mathcal{F}$.
A subset M of $\mathcal{F}$ is called ϱ-complete iff $[M]$ is a ϱ-algebra.

Definition: Let ϱ be an equivalence relation on $\mathcal{F}$.
A subalgebra M of $\mathcal{F}$ is called ϱ-maximal iff the following conditions are satisfied:
– M is not a ϱ-algebra and
– any subalgebra N with $M \subset N$ is a ϱ-algebra.

ϱ-Completeness – Example

Equivalence relation ϱ_1 :
$\varrho_1 = \{(F, G) \mid arity(F) = arity(G), F((0, 0, \ldots, 0)) = G((0, 0, \ldots, 0))\}$

Equivalence classes of ϱ_1 :
$K_{n,a} = \{F \mid F$ has arity $n, F((0, 0, \ldots, 0)) = a\}$, $\quad n \in \mathbf{N}$, $a \in \{0, 1\}$

ϱ_1-algebras: $\mathcal{F}$,
$\{F \mid F_p(q) = 0^{|q|}$ for $|p| \geq 1\}$

ϱ_1-completeness:
M is ϱ_1-complete iff M contains at least one F with $F((0, 0, \ldots, 0)) = 0$ and at least one G with $G((0, 0, \ldots, 0)) = 1$.

ϱ_1-maximal subalgebra: $M_{T_0} = \{F \mid F((0, 0, \ldots, 0)) = 0\}$

Congruence Relations

complete relation – $\sigma_C = \{(F,G) \mid F,G \in \mathcal{F}\}$

equality relation – $\sigma_E = \{(F,G) \mid F = G\}$

arity relation – $\sigma_A = \{(F,G) \mid arity(F) = arity(G)\}$

t-bounded equality – $\sigma_t = \{(F,G) \mid F(p) = G(p)$ for all p with $|p| \leq t\}$
$t \in \mathbf{N}$

Theorem:

i) σ_C, σ_E, σ_A and σ_t for $t \geq 1$ are the only congruence relations on $\mathcal{F}'$.

ii) σ_C, σ_E, σ_A and σ_t for $t \geq 1$ are the only congruence relations on $\mathcal{F}$.

Completeness with respect to the complete and arity relation

Results for σ_C:

– Any non-empty subalgebra of $\mathcal{F}$ is a σ_C-algebra.

– Any non-empty subset of $\mathcal{F}$ is σ_C-complete.

– $\emptyset$ is the only σ_C-maximal subalgebra.

– The σ_C-completeness of a finite subset of $\mathcal{F}$ is decidable.

Results for σ_A:

– Any non-empty subalgebra of $\mathcal{F}$ is a σ_A-algebra.

– Any non-empty subset of $\mathcal{F}$ is σ_A-complete.

– $\emptyset$ is the only σ_A-maximal algebra.

– The σ_A-completeness of a finite subset of $\mathcal{F}$ is decidable.

Completeness with respect to equality relation

σ_E-completeness is the "classical" completeness

- For any n, there is a complete set $M = \{F_1, F_2, \ldots, F_n\}$ such that $M \setminus \{F_i\}$ is not complete.
- M is complete if and only if M is not contained in any maximal subalgebra.
- The cardinality of the set of maximal subalgebras is the cardinality of the set of real numbers.
- There is a countable set N of maximal subalgebras such that $M \subset \mathcal{F}$ is complete iff M is not contained in any algebra of N.
- There is no algorithm to decide the completeness of a finite subset of $\mathcal{F}$.

Completeness with respect to t-bounded equality I

Theorem:
For any set $M \subset \mathcal{F}$ and any $t \in \mathbf{N}$,
$[M]$ and $< M >$ are σ_t-equivalent,
i.e., for any two functions $F \in [M]$ and $F' \in < M >$, there are functions $G \in < M >$ and $G' \in [M]$ such that $(F, G) \in \sigma_t$ and $(F', G') \in \sigma_t$.

Theorem:
For any $t \in \mathbf{N}$, there is a mapping $\tau_t : \mathcal{F} \to P_{2^t}$ such that
$M \subset \mathcal{F}$ is σ_t-complete if and only if $\tau_t(M) = \{\tau_t(F) \mid F \in M\}$ generates $\tau_t(\mathcal{F})$.

Completeness with respect to t-bounded equality II

Theorem: Let $t \in \mathbf{N}$. $M \subseteq \mathcal{F}$ is σ_t-complete if and only if M is not contained in any σ_t-maximal subalgebra of $\mathcal{F}$.

Theorem: For any $t \in \mathbf{N}$, there is a finite number of σ_t-maximal subalgebras.

There is a description of all σ_t-maximal subalgebras by relations
(F preserves R iff $(p,q) \in R$ implies $(F(p),F(q)) \in R$,
$U(R)$ – set of functions preserving R,
for any σ_t-maximal subalgebra M, there is a relation R with $M = U(R)$)

Theorem For any $t \in \mathbf{N}$, the σ_t-completeness of a finite subset of $\mathcal{F}$ is decidable.

Regular Sets

product	–	$L \cdot L' = \{ww' \mid w \in L,\ w' \in L'\}$
power	–	$L^0 = \{\lambda\}$, $L^{i+1} = L \cdot L^i$ for $i \geq 0$,
Kleene-closure	–	$L^* = \bigcup_{i \geq 0} L^i$

Definition:
A set L of words over X is called regular iff L can be obtained from $\emptyset$, $\{\lambda\}$ and $\{x\}$ where $x \in X$ by iterated application of union, product and Kleene-closure.

Example: If $1 \in X$, then $X^*\{1\}\{1\}$ is regular.

Sequential Functions as Acceptors

Definition:
Let $F : X^* \to Y^*$ be a sequential function and $\emptyset \subset Y' \subset Y$. Then the language $T(F, Y')$ accepted by F and Y' is defined as the set of all words p such that $F(p) = p'y$ with $y \in Y'$.

Example: F from Foil 5,

$$\begin{aligned} T(F, \{1\}) &= \text{set of all words ending with two 1's} \\ &= \{0,1\}^*\{1\}\{1\} \end{aligned}$$

Theorem:
A set $L \subset X^*$ is regular if and only if there are a sequential function $F : X^* \to Y^*$ and a set Y', $\emptyset \subset Y' \subset Y$, such that $T(F, Y') = L$.

Kleene-Equivalence

Definition:
Two sequential functions $F : X^* \to Y^*$ and $G : X^* \to Z^*$ are Kleene-equivalent (written as $(F, G) \in \sigma_K$) if and only if there are sets Y' and Z' with $\emptyset \subset Y' \subset Y$ and $\emptyset \subset Z' \subset Z$ such that $T(F, Y') = T(G, Z')$.

Lemma:
σ_K is an equivalence relation on $\mathcal{F}$.

$F_{(non)} \in \mathcal{F}^1$ defined by
$F_{(non)}(x_1 x_2 \ldots x_n) = non(x_1)\ non(x_2) \ldots\ non(x_n)$

Theorem: Let $F \in \mathcal{F}^m$ and $G \in \mathcal{F}^m$.
$(F, G) \in \sigma_K$ if and only if $F = F_{(non)} \circ G$ (and $G = F_{(non)} \circ F$).

Completeness with respect to Kleene-Equivalence

For $i \in \{0, 1\}$, $M_{T_i} = \{F \mid F((i, i, \ldots, i)) = i\}$

Theorem: The only σ_K-algebras of $\mathcal{F}$ are M_{T_0}, M_{T_1} and $\mathcal{F}$.

Theorem: There is no algorithm which decides the σ_K-completeness of a finite set.

Theorem:
i) $M \subset \mathcal{F}$ is σ_K-complete if and only if M is not contained in any σ_K-maximal subalgebra.
ii) The cardinality of the set of σ_K-maximal subalgebras of $\mathcal{F}$ is the cardinality of the set of real numbers.
iii) There is a countable set N of σ_K-maximal subalgebras of $\mathcal{F}$ such that $M \subset \mathcal{F}$ is complete iff M is not contained in any algebra of N.

Variations of Kleene-Equivalence I

Definition:

i) $F \in \mathcal{F}^m$ and $G \in \mathcal{F}^m$ are called negation-equivalent (written as $(F,G) \in \sigma_N$) if and only if

$$F(p_1, p_2, \ldots, p_n) = G(F_{(non)}(p_1), F_{(non)}(p_2), \ldots, F_{(non)}(p_n)) .$$

ii) $F \in \mathcal{F}^m$ and $G \in \mathcal{F}^m$ are called dual (written as $(F,G) \in \sigma_D$) if and only if

$$F(p_1, p_2, \ldots, p_n) = F_{(non)}(G(F_{(non)}(p_1), F_{(non)}(p_2), \ldots, F_{(non)}(p_n))) .$$

Variations of Kleene-Equivalence II

Theorem:

i) $M \subset \mathcal{F}$ is σ_N-complete if and only if M is complete.

ii) $M \subset \mathcal{F}$ is σ_D-complete if and only if M is complete.

Corollary:

i) There is no algorithm which decides the σ_N-completeness of a finite set.

ii) There is no algorithm which decides the σ_D-completeness of a finite set.

Metric Equivalence – Definition

Definition:
For $F \in \mathcal{F}^m$ and $G \in \mathcal{F}^m$, we define

$$d(F,G) = \frac{1}{t} \quad \text{iff} \quad \begin{array}{l} F(p) = G(p) \text{ for all } p \text{ with } |p| \leq t-1 \text{ and} \\ F(p') \neq G(p') \text{ for some } p' \text{ with } |p'| = t \end{array}$$

Definition:
$M \subseteq \mathcal{F}$ and $M' \subseteq \mathcal{F}$ are called metrically equivalent if, for any $t \in \mathbf{N}$ and any two sequential functions $F \in M$ and $F' \in M'$, there are sequential functions $G \in M'$ and $G' \in M$ such that

$$d(F,G) \leq \frac{1}{t} \quad \text{and} \quad d(F',G') \leq \frac{1}{t}\,.$$

Metric Equivalence – Example

For $F \in \mathcal{F}$ and $i \in \mathbf{N}$, let F_i be defined by

$$F_i(p) = F(p) \quad \text{for} \quad p \text{ with } |p| \leq i$$

$$F_i(q) = F(q')0^n \quad \text{for} \quad q \text{ with } q = q'q'', |q'| = i, |q''| = n \geq 0$$

$Q = \{F_i \mid F \in \mathcal{F},\ i \in \mathbf{N}\}$ is metrically equivalent to $\mathcal{F}$

Note that Q is a subalgebra,
$Q \subset \mathcal{F}$, and
Q is not finitely generated.

Metric Completeness

Definition:

$M \subset \mathcal{F}$ is called metrically complete if and only if $[M]$ is metrically equivalent to $\mathcal{F}$.

Theorem:

There is no algorithm which decides whether or not a finite set is metrically complete.

Metrically Maximal Subalgebras

Definition:
A subalgebra $M \subset \mathcal{F}$ is called metrically maximal if
– M is not metrically complete and
– $M \cup \{F\}$ is metrically complete for any $F \in \mathcal{F}$.

Theorem:
i) M is metrically complete if and only if M is not contained in any metrically maximal subalgebra of $\mathcal{F}$.
ii) Any metrically maximal subalgebra of $\mathcal{F}$ is σ_t-maximal for some $t \in \mathbf{N}$. For $t \in \mathbf{N}$, any σ_t-maximal subalgebra of $\mathcal{F}$ is a metrically maximal subalgebra.
iii) The cardinality of the set of all metrically maximal subalgebras of $\mathcal{F}$ is the cardinality of $\mathbf{N}$.