COUNTING OCCURRENCES OF A PATTERN OF LENGTH THREE WITH AT MOST TWO DISTINCT LETTERS IN A k-ARY WORD

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ABSTRACT

Define \( \tau(\pi) \) to be the number of subsequences of \( \pi \) that are order-isomorphic to \( \tau \). Let \( \tau \) be a pattern of length three with at most two distinct letters, namely, \( \tau \in \{111, 112, 121, 122, 211, 212, 221\} \).

In this paper, we give an algorithm for finding the generating function \( w_{\tau,r}(n;y) = \sum_{k \geq 1} \sum_{\pi \in [k]^n, \tau(\pi) = r} y^k \) for the number of \( k \)-ary words of length \( n \) that contain exactly \( r \) occurrences of the pattern \( \tau \), for given \( r \geq 0 \). In particular, we obtain explicit formulas for the generating functions \( w_{\tau,r}(n;y) \), where \( r = 0, 1 \).

Keywords: \( k \)-ary word, pattern, enumeration, generating function, Eulerian polynomial

1. Introduction

Permutations. We denote the set of permutations of \([n] = \{1, 2, \ldots, n\}\) by \( S_n \). We shall view permutations in \( S_n \) as words with \( n \) distinct letters in \([n]\). A permutation or just pattern is a permutation \( \tau \in S_k \), and an occurrence of \( \tau \) in a permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \in S_n \) is a subsequence of \( \pi \) that is order-isomorphic to \( \tau \). For instance, an occurrence of 312 is a subsequence \( \pi \) of \( \pi_1 \pi_2 \pi_3 \) such that \( \pi_1 < \pi_3 < \pi_2 \). We denote the number of permutations in \( S_n \) that contain exactly \( r \) occurrences of the pattern \( \tau \) by \( s_{\tau,r}(n) \). In the last two decades much attention has been paid to the problem of finding the numbers \( s_{\tau,r}(n) \) for a fixed \( r \geq 0 \) and a given pattern \( \tau \) (see [1][2][8][10][12] and references therein). Up to now, only the