References I

F. Drewes: Grammatical Picture Generation. Springer-Verlag, 2006.

H. Freeman, Computer processing of line-drawing images. Computer Surveys 6 (1974) 57-97.

H. W. Maurer, G. Rozenberg, E. Welzl: Using string languages to describe picture languages. Inform. Control 54 (1982) 155-185.

I. H. Sudborough and E. Welzl: Complexity and decidability of chain code picture languages. Theor. Comp. Sci. 36 (1985) 175-202.

J. Dassow, F. Hinz: Decision problems and regular chain code picture languages. Discrete Appl. Math. 45 (1993) 29-49

P. Pruzinkiewicz, A. Lindenmayer: The Algorithmic Beauty of Plants. Springer, New York, 1990.

P. Pruzinkiewicz, M. Hammel, J. Hanan, R. Mech: Visual models of plant development. In: Handbook of Formal Languages, Vol. III, Springer, Berlin, 1997, 535–597.

Descriptions of a Chain Code Picture

Definition:
For a drawn chain code picture q and and basic chain code picture p, we set

$$des(q) = \{w \in \pi^* \mid dccp(w) = q\}$$

and

$$des(p) = \{w \in \pi^* \mid bccp(w) = p\}$$

.

Theorem:
For a drawn chain code picture q and and basic chain code picture p, $des(q)$ and $des(p)$ are regular languages.

Operations red and ref I

$R = \{ud, du, lr, rl\}$ — set of retreats

A word w is called *retreat-free*, if no word of R is a subword of w.

Definition:
For a word $w \in \pi^*$, we define the *retreat deletion image* $red(w)$ inductively as follows:
(1) $w \in red(w)$,
(2) If $z \in red(w)$ and $z = z_1 s z_2$ for some $s \in R$, then $z_1 z_2 \in red(w)$.
(3) A word belongs to $red(w)$ if and only if it is constructed by steps (1) or (2).

For a language $L \subseteq \pi^*$, we set $red(L) = \bigcup_{w \in L} red(w)$.

Operations red and ref II

Lemma: Let $w \in \pi^*$.
— Then $red(w)$ contains exactly one retreat-free word.
— For any word $z \in red(w)$, $sh(z) = sh(w)$.

For $w \in \pi^*$, let $ref(w)$ be the only retreat-free word in $red(w)$.

For a language $L \subseteq \pi^*$, we set $ref(L) = \{ref(w) \mid w \in L\}$.

Lemma: $ref(\pi^*)$ is a regular language.

Lemma: $D_\pi = \{w \mid ref(w) = \lambda\}$ is context-free.

Theorem: i) If $L \subseteq \pi^*$ is a regular language, then $red(L)$ and $ref(L)$ are regular, too (i.e., $\mathcal{L}(REG)$ is closed under the operations red and ref).
ii) $\mathcal{L}(CF)$ and $\mathcal{L}(CS)$ are not closed under red and ref.

Grammars Generating Pictures

$dccp(G) = \{dccp(w) \mid w \in L(G)\}$ and $bccp(G) = \{bccp(w) \mid w \in L(G)\}$

$$G_1 = (\{S\},\ \pi,\ \{S \rightarrow urdluS,\ S \rightarrow urdlu\}, S)$$

$$G_2 = (\{S, A\},\ \pi,\ \{S \rightarrow rAr,\ A \rightarrow uAd,\ A \rightarrow ud\},\ S)$$

$$G_3 = ((\{S, A\},\ \pi,\ \{S \rightarrow rAr,\ A \rightarrow uAd,\ A \rightarrow r\},\ S)$$

$$G_4 = (\{S.A\},\ \pi,\ \{S \rightarrow lA,\ A \rightarrow urlAd,\ A \rightarrow urd\},\ S)$$

$$G_5 = (\{S, A\},\ \pi,\ \{S \rightarrow SrS,\ S \rightarrow lA,\ A \rightarrow urlAd,\ A \rightarrow urd\},\ S)$$

$$G_6 = (\{S\},\ \pi,\ \{S \rightarrow urdlS,\ S \rightarrow ldruS,\ S \rightarrow \lambda\},\ S)$$

CHOMSKY Hierarchy for Chain Code Picture Languages

Definition: A chain code picture language B is called *regular* or *context-free* or *monotone* or *recursively enumerable* if there is a regular or context-free or monotone grammar or a phrase structure grammar, respectively, such that $B = bccp(L(G))$.

By $\mathcal{CCP}(REG)$, $\mathcal{CCP}(CF)$, $\mathcal{CCP}(CS)$, and $\mathcal{CCP}(RE)$, we denote the families of all regular, context-free, monotone and recursively enumerable chain code picture languages.

Theorem:
$\mathcal{CCP}(REG) \subset \mathcal{CCP}(CF) \subset \mathcal{CCP}(CS) = \mathcal{CCP}(RE)$

Membership Problem for Chain Code Picture Languages

(1) *Membership problem*: Given a grammar $G = (N, \pi, P, S)$
and a chain code picture p
Decide whether or not $p \in bccp(G)$.

(2) Given a grammar $G = (N, \pi, P, S)$ and a chain code picture p
Decide whether the set $\{w \mid w \in L(G)$ and $bccp(w) = p\}$ is finite.

(3) Given a grammar $G = (N, \pi, P, S)$ and a word $w \in \pi^*$
Decide whether the set $\{w \mid w \in L(G)$ and $bccp(w) = p\}$ is a singleton.

Theorem: i) The problems (1), (2), and (3) are decidable for context-free grammars G.
ii) The problems (1), (2), and (3) are undecidable for monotone grammars G.

Theorem: The membership problem is **NP**-complete for regular grammars.

Emptiness and Finiteness Problems for Chain Code Picture Languages

Theorem: The emptiness problem

Given a phrase structure grammar $G = (N, \pi, P, S)$,
decide whether or not the picture set $bccp(G)$ is empty?

is decidable for context-free grammars and undecidable for monotone grammars.

Theorem: The finiteness problem

Given a phrase structure grammar $G = (N, \pi, P, S)$,
decide whether or not the picture set $bccp(G)$ is finite

is decidable for context-free grammars and undecidable for monotone grammars.

Normal Context-Free Grammars

Definition: A context-free grammar $G = (N, \pi, P, S)$ is called normal if, for every nonterminal $A \in N$ and any derivation $A \Longrightarrow^* xAy$ with $x, y \in \pi^*$, $sh(x) = sh(y) = (0, 0)$.

Lemma: For any normal context-free grammar G, there exist a constant c such that, for any $w \in L(G)$, $\sqrt{m^2 + n^2} \leq c$ holds where $(m, n) = sh(w)$.

Lemma: Let $G = (N, \pi, P, S)$ be a normal context-free grammar. Then there exist a normal context-free grammar $G' = (N', \pi, P', S')$ such that $L(G') = pref(L(G))$.

Corollary: Let G be a context-free grammar. Then $bccp(G)$ is finite if and only if G is normal.

Lemma: It is decidable whether or not a given context-free grammar G is normal.

Equivalence Problem for Chain Code Picture Languages

Theorem:
The equivalence problem

> Given two phrase structure grammars $G_1 = (N_1, \pi, P_1, S_1)$
> and $G_2 = (N_2, \pi, P_2, S_2)$
> decide whether or not $bccp(G_1) = bccp(G_2)$ holds?

is undecidable for regular grammars.

Subpicture Problem for Chain Code Picture Languages

Definition:
We say that the basic chain code picture p is a *subpicture* of the basic chain code picture q if there is a chain code picture p' such that $p' \equiv p$ and $p' \subseteq q$.
We say that the basic chain code picture p is a subpicture of the basic chain code picture language L, if p is a subpicture of some $q \in L$.

Theorem:
i) For an arbitrary basic chain code picture p and an arbitrary context-free grammar $G = (N, \pi, P, S)$, it is decidable whether or not p is a subpicture of $bccp(G)$.
i) For an arbitrary chain code picture p and an arbitrary monotone grammar $G = (N, \pi, P, S)$, it is undecidable whether or not p is a subpicture of $bccp(G)$.

Universal Subpicture Problem for Chain Code Picture Languages

Definition:
We say that the basic chain code picture p is a $universal\ subpicture$ of the basic chain code picture language L, if p is a subpicture of any $q \in L$.

Theorem:
For an arbitrary basic chain code picture p and an arbitrary regular grammar $G = (N, \pi, P, S)$, it is undecidable whether or not p is a universal subpicture of $bccp(G)$.

Some "geometric" properties

A chain code picture p is a *simple* curve, if all its nodes have a degree at most 2.
A chain code picture p is a *closed simple* curve, if all its nodes have degree 2.

A chain code picture p is a *tree*, if it does not contain a closed simple curve as a subpicture.

A chain code picture p is called *regular*, if all nodes of p have the same degree.

A chain code picture p is called *Eulerian*, if
— all nodes of p have an even degree or
— there are two nodes n and n' in p such that all nodes of p different from n and n' have even degree.

A chain code picture p is called *Hamiltonian*, if it contains a subpicture p which is a simple curve and contains all nodes of p.

Decidability of "geometrical" properties I

Theorem:
Given a regular grammar $G = (N, \pi, P, S)$, it is undecidable whether or not $bccp(G)$ contains

a) a simple curve,

b) a closed simple curve,

c) a Eulerian picture,

d) a tree

e) a Hamiltonian picture,

f) a regular picture.

Decidability of "geometrical" properties II

Theorem: For an arbitrary context-free grammar $G = (N, \pi, P, S)$, it is decidable whether or not all pictures of $bccp(G)$ are trees.

Lemma: Let G be a regular grammar such that all elements of $bccp(G)$ are closed simple curves. Then $bccp(G)$ is finite.

A chain code picture p is called *convex* if there is a chain code picture q such that $p \cup q$ is a closed simple curve and the intersection of the inner part of $p \cup q$ with any straight line which is parallel to one of the axes is a finite straight line.

Theorem: For an arbitrary regular grammar $G = (N, \pi, P, S)$, it is decidable whether or not

a) all pictures of $bccp(G)$ are rectangles,

b) $bccp(G)$ contains a rectangle,

c) $bccp(G)$ contains a convex picture.