Error Types I

Definition:
An exchange error is the transmission of 0 instead of 1 or the transmission of 1 instead of 0.
A deletion error is the deletion of a symbol during a transmission (the transmitted word is shorter).
An insertion error is the transmission of an extra symbol (the transmitted word is longer).

Notation: $1 \to 0,\ 0 \to 1,\ 0 \to \lambda,\ 1 \to \lambda,\ \lambda \to 0,\ \lambda \to 1$

Error Types II

$G = \{1 \to 0,\ 0 \to 1,\ 0 \to \lambda,\ 1 \to \lambda,\ \lambda \to 0,\ \lambda \to 1\}$

Definition:
We call a subset of G an error type.
An error type F is called symmetrical, if F is obtained by some union of the sets $\{0 \to 1, 1 \to 0\}$, $\{\lambda \to 0, 0 \to \lambda\}$, and $\{\lambda \to 1, 1 \to \lambda\}$.

For an error type F and words w and v over $\{0, 1\}$, we set

$$w \overset{F,t}{\Longrightarrow} v,$$

if the word v arises from the word w during the transmission by the simutaneous occurrence of t errors from F.

Error Correction I

Definition:
Let F be an error type and C be a block code over $\{0,1\}$. Then we call C a code with correction of s errors from F, if, for any word $v \in \{0,1\}^*$, there is at most one word $w \in C$ with $w \stackrel{F,t}{\Longrightarrow} v$ and $t \leq s$.

For an error type F and words $w, v \in \{0,1\}^*$, we define

$$d_F(w,v) = \begin{cases} \min\{t \mid w \stackrel{F,t}{\Longrightarrow} v\}, & \text{if this exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem:
For a symmetrical error type F, a metric function on $\{0,1\}^*$ is given by d_F.

Error Correction II

Definition:
For a symmetrical error type F and a finite code C, we define the code distance $d_F(C)$ as

$$d_F(C) = \min\{d_F(x,y) \mid x,y \in C, x \neq y\}.$$

Theorem:
Let F be a symmetrical error type. Then, a finite code C is a code with correction of s errors from F if and only if

$$d_F(C) \geq 2s+1$$

holds.

Estimations I

For natural numbers $n \geq 1$ and $d \geq 1$, we set

$$m(n,d) = \max\{\#(C) \mid C \subseteq \{0,1\}^n, d(C) \geq d\}\,.$$

Theorem:
For $n \geq 3$ and $s \geq 1$, we have

$$\frac{2^n}{\sum_{k=0}^{2s} \binom{n}{k}} \leq m(n, 2s+1) \leq \frac{2^n}{\sum_{k=0}^{s} \binom{n}{k}}\,.$$

Estimations II

Theorem: For two positive natural numbers n and d (with $n \geq d$), we have

$$m(n, d) \leq 2 \cdot m(n-1, d) .$$

Theorem: Let n and d be two positive natural numbers (with $n \geq d$).
i) Then $m(n, d) \geq m(n+1, d+1)$.
ii) If d is odd, then $m(n, d) = m(n+1, d+1)$.

Theorem: For two positive natural numbers n and d (with $n \geq d$), we have

$$m(2n, 2d) \geq m(n, d) \cdot m(n, 2d) .$$

Estimations III

Theorem: Let n and d be two positive natural numbers n and d (with $n \geq d$).

i) If d is even, we have

$m(n,d) \leq 2 \cdot \lfloor \frac{d}{2d-n} \rfloor$ for $2d > n$,

$m(n,d) \leq 2n$ for $2d = n$.

ii) If d is odd, we have

$m(n,d) \leq 2 \cdot \lfloor \frac{d+1}{2d+1-n} \rfloor$ for $2d+1 > n$,

$m(n,d) \leq 2n$ for $2d+1 = n$.

iii) For $n \geq 2d$, we have

$m(n,d) \leq d \cdot 2^{n-2d+2}$ for even d,

$m(n,d) \leq (d+1) \cdot 2^{n-2d+1}$ for odd d.