Logik für Informatiker
Logic for computer scientists

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Proofs for Boolean Logic

Logical consequence

1. Q is a logical consequence of $P_1, \ldots, P_n$, if all worlds that make $P_1, \ldots, P_n$ true also make Q true.
2. Q is a tautological consequence of $P_1, \ldots, P_n$, if all valuations of atomic formulas with truth values that make $P_1, \ldots, P_n$ true also make Q true.
3. Q is a TW-logical consequence of $P_1, \ldots, P_n$, if all worlds from Tarski's world that make $P_1, \ldots, P_n$ true also make Q true.

The difference lies in the set of worlds that is considered:

1. all worlds (whatever this exactly means . . .)
2. all valuations of atomic formulas with truth values ($=$ rows in the truth table)
3. all block worlds from Tarski's world

Proofs

- With proofs, we try to show (tauto)logical consequence
- Truth-table method can lead to very large tables, proofs are often shorter
- Proofs are also available for consequence in full first-order logic, not only for tautological consequence

Limits of the truth-table method

1. truth-table method leads to exponentially growing tables
   - 20 atomic sentences $\Rightarrow$ more than 1.000.000 rows
2. truth-table method cannot be extended to first-order logic

- model checking can overcome the first limitation (up to 1.000.000 atomic sentences)
- proofs can overcome both limitations

Proofs

- A proof consists of a sequence of proof steps
- Each proof step is known to be valid and should
    - be significant but easily understood, in informal proofs,
    - follow some proof rule, in formal proofs.
- Some valid patterns of inference that generally go unmentioned in informal (but not in formal) proofs:
    - From $P \wedge Q$, infer P.
    - From P and Q, infer $P \wedge Q$.
    - From P, infer $P \vee Q$.

Proof by cases (disjunction elimination)

To prove S from $P_1 \vee \ldots \vee P_n$, prove S from each of $P_1, \ldots, P_n$.

Claim: there are irrational numbers b and c such that b^c is rational.

Proof: $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.

Case 1: If $\sqrt{2}^{\sqrt{2}}$ is rational: take $b = c = \sqrt{2}$.

Case 2: If $\sqrt{2}^{\sqrt{2}}$ is irrational: take $b = \sqrt{2}^{\sqrt{2}}$ and $c = \sqrt{2}$.

Then $b^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2$.

Proof by contradiction

To prove $\neg S$, assume S and prove a contradiction $\perp$.
($\perp$ may be infered from P and $\neg P$.)
Assume $Cube(c) \lor Dodec(c)$ and $Tet(b)$.
Claim: $\neg(b = c)$.
Proof: Let us assume $b = c$.
Case 1: If $Cube(c)$, then by $b = c$, also $Cube(b)$, which contradicts $Tet(b)$.
Case 2: $Dodec(c)$ similarly contradicts $Tet(b)$.
In both case, we arrive at a contradiction. Hence, our assumption $b = c$ cannot be true, thus $\neg(b = c)$.

Arguments with inconsistent premises

A proof of a contradiction $\perp$ from premises $P_1, \ldots, P_n$ (without additional assumptions) shows that the premises are inconsistent. An argument with inconsistent premises is always valid, but more importantly, always unsound.

$$
\begin{array}{|l}
\text{Home(max)} \lor \text{Home(claire)} \\
\neg\text{Home(max)} \\
\neg\text{Home(claire)} \\
\hline
\text{Home(max)} \land \text{Happy(carl)}
\end{array}
$$

Arguments without premises

A proof without any premises shows that its conclusion is a logical truth.

Example: $\neg(P \wedge \neg P)$.

Formal Proofs and Boolean Logic

Formal proofs in Fitch

- Well-defined set of formal proof rules
- Formal proofs in Fitch can be mechanically checked
- For each connective, there is
  - an introduction rule, e.g. "from P, infer $P \vee Q$".
  - an elimination rule, e.g. "from $P \wedge Q$, infer P".

Conjunction Elimination ($\wedge$ Elim)

$$\begin{array}{l|l} & \mathsf{P}_1 \wedge \ldots \wedge \mathsf{P}_i \wedge \ldots \wedge \mathsf{P}_n \\ & \vdots \\ \triangleright & \mathsf{P}_i \end{array}$$

Conjunction Introduction ($\wedge$ Intro)

$$
\begin{array}{l|l}
 & P_1 \\
 & \Downarrow \\
 & P_n \\
 & \vdots \\
\triangleright & P_1 \wedge \ldots \wedge P_n
\end{array}
$$

Disjunction Introduction (∨ Intro)

$$
\begin{array}{l|l}
 & P_i \\
 & \vdots \\
\triangleright & P_1 \lor \ldots \lor P_i \lor \ldots \lor P_n
\end{array}
$$

Disjunction Elimination (∨ Elim)

$$
\begin{array}{l|l}
 & P_1 \vee \ldots \vee P_n \\
 & \vdots \\
 & \quad \begin{array}{|l} P_1 \\ \hline \vdots \\ S \end{array} \\
 & \quad \Downarrow \\
 & \quad \begin{array}{|l} P_n \\ \hline \vdots \\ S \end{array} \\
 & \vdots \\
\triangleright & S
\end{array}
$$

The proper use of subproofs

1. $(B \wedge A) \vee (A \wedge C)$	
2. $B \wedge A$	
3. B	$\wedge$ **Elim**: 2
4. A	$\wedge$ **Elim**: 2
5. $A \wedge C$	
6. A	$\wedge$ **Elim**: 5
7. A	$\vee$ **Elim**: 1, 2–4, 5–6
8. $A \wedge B$	$\wedge$ **Intro**: 7, 3

The proper use of subproofs (cont'd)

- In justifying a step of a subproof, you may cite any earlier step contained in the main proof, or in any subproof whose assumption is still in force. You may never cite individual steps inside a subproof that has already ended.
- Fitch enforces this automatically by not permitting the citation of individual steps inside subproofs that have ended.

⊥ Introduction (⊥ Intro)

$$
\begin{array}{l|l}
 & P \\
 & \vdots \\
 & \neg P \\
 & \vdots \\
\triangleright & \bot
\end{array}
$$

Negation Introduction (¬ Intro)

		P
		⋮
		$\bot$
$\triangleright$	$\neg$P	

Negation Elimination (¬ Elim)

$$
\begin{array}{l|l}
 & \neg\neg P \\
 & \vdots \\
\triangleright & P
\end{array}
$$

$\perp$ Elimination ($\perp$ Elim)

$$
\begin{array}{c|l}
 & \perp \\
 & \vdots \\
\triangleright & P
\end{array}
$$

Strategies and tactics in Fitch

1. Understand what the sentences are saying.
2. Decide whether you think the conclusion follows from the premises.
3. If you think it does not follow, or are not sure, try to find a counterexample.
4. If you think it does follow, try to give an informal proof.
5. If a formal proof is called for, use the informal proof to guide you in finding one.
6. In giving consequence proofs, both formal and informal, don't forget the tactic of working backwards.
7. In working backwards, though, always check that your intermediate goals are consequences of the available information.

Strategies in Fitch, cont'd

- Always try to match the situation in your proof with the rules in the book (see book appendix for a complete list)
- Look at the main connective in a premise, apply the corresponding elimination rule (forwards)
- Or: look at the main connective in the conclusion, apply the corresponding introduction rule (backwards)