Logik für Informatiker
Logic for computer scientists

Till Mossakowski

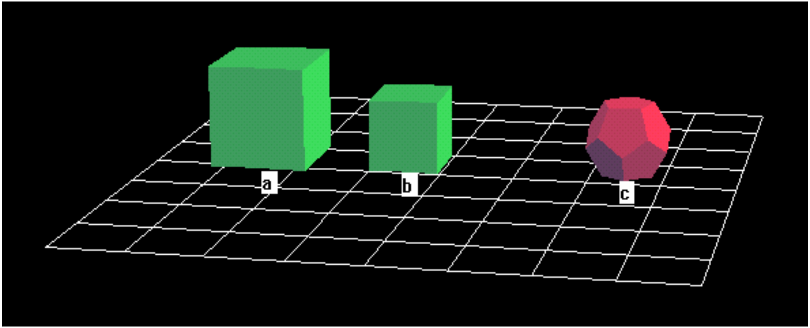
WiSe 2013/14

First-order structures

First-order structures: motivation

- How to make the notion of logical consequence more formal?
- For propositional logic: truth tables $\Rightarrow$ tautological consequence
- For first-order logic, we need also to interpret quantifiers and identity
- The notion of first-order structure models Tarski's world situations and real-world situations using set theory

Example

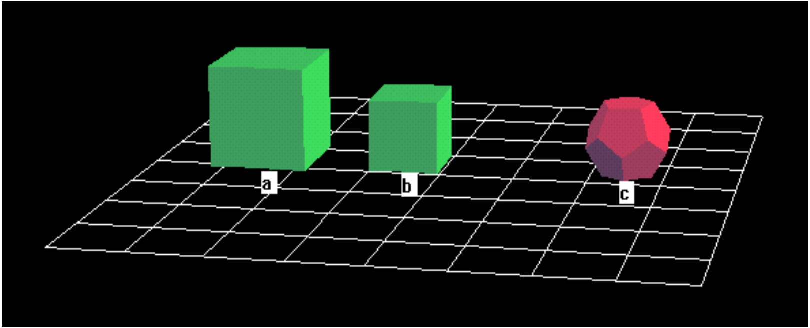


First-order structures: definition

A first-order structure $\mathfrak{M}$ consists of:

- a nonempty set $D^{\mathfrak{M}}$, the domain of discourse;
- for each n-ary predicate P of the language,
  a set $\mathfrak{M}(P)$ of n-tuples $\langle x_1, \ldots, x_n \rangle$ of elements of $D^{\mathfrak{M}}$,
  called the extension of P.
  The extension of the identity symbol $=$ must be
  $\{\langle x, x \rangle \mid x \in D^{\mathfrak{M}}\}$;
- for any name (individual constant) c of the language, an element $\mathfrak{M}(c)$ of $D^{\mathfrak{M}}$.

Example



An interpretation according to Tarski's World

Assume the language consists of the predicates *Cube*, *Dodec* and *Larger* and the names *a*, *b* and *c*.

- $D^{\mathfrak{M}} = \{Cube1, Cube2, Dodec1\}$;
- $\mathfrak{M}(Cube) = \{Cube1, Cube2\}$;
- $\mathfrak{M}(Dodec) = \{Dodec1\}$;
- $\mathfrak{M}(Larger) = \{(Cube1, Cube2), (Cube1, Dodec1)\}$;
- $\mathfrak{M}(=) =$
  $\{(Cube1, Cube1), (Cube2, Cube2), (Dodec1, Dodec1)\}$;
- $\mathfrak{M}(a) = Cube1$; $\mathfrak{M}(b) = Cube2$; $\mathfrak{M}(c) = Dodec1$.

An interpretation not conformant with Tarski's World

- $D^{\mathfrak{M}} = \{Cube1, Cube2, Dodec1\}$;
- $\mathfrak{M}(Cube) = \{Dodec1, Cube2\}$;
- $\mathfrak{M}(Dodec) = \emptyset$;
- $\mathfrak{M}(Larger) = \{(Cube1, Cube1), (Dodec1, Cube2)\}$;
- $\mathfrak{M}(=) = \{(Cube1, Cube1), (Cube2, Cube2), (Dodec1, Dodec1)\}$;
- $\mathfrak{M}(a) = Cube1$; $\mathfrak{M}(b) = Cube2$; $\mathfrak{M}(c) = Dodec1$.

Variable assignments

A variable assignment in $\mathfrak{M}$ is a (possibly partial) function g defined on a set of variables and taking values in $D^{\mathfrak{M}}$.
Given a well-formed formula P, we say that the variable assignment g is appropriate for P if all the free variables of P are in the domain of g, that is, if g assigns objects to each free variable of P.

Examples

$D^{\mathfrak{M}} = \{Cube1, Cube2, Dodec1\}$
g_1 assignes $Cube1$, $Cube2$, $Dodec1$ to the variables x, y, z, respectively.
g_1 is appropriate for $Between(x, y, z) \wedge \exists u(Large(u))$, but not for $Larger(x, v)$.

g_2 is the empty assignment.
g_2 is only appropriate for well-formed formulas without free variables (that is, for sentences).

Variants of variable assignments

If g is a variable assignment, $g[v/b]$ is the variable assignment

- whose domain is that of g plus the variable v, and
- which assigns the same values as g, except that
- it assigns b to the variable v.

$[t]_g^{\mathfrak{M}}$

$[t]_g^{\mathfrak{M}}$ is

- $\mathfrak{M}(t)$ if t is an individual constant, and
- $g(t)$ if t is a variable.

Satisfaction (A. Tarski)

1. $\mathfrak{M} \models R(t_1, \ldots, t_n)[g]$ iff $\langle [t_1]_g^{\mathfrak{M}}, \ldots, [t_n]_g^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$;
2. $\mathfrak{M} \models \neg P[g]$ iff it is not the case that $\mathfrak{M} \models P[g]$;
3. $\mathfrak{M} \models P \wedge Q[g]$ iff both $\mathfrak{M} \models P[g]$ and $\mathfrak{M} \models Q[g]$;
4. $\mathfrak{M} \models P \vee Q[g]$ iff $\mathfrak{M} \models P[g]$ or $\mathfrak{M} \models Q[g]$ or both;
5. $\mathfrak{M} \models P \rightarrow Q[g]$ iff not $\mathfrak{M} \models P[g]$ or $\mathfrak{M} \models Q[g]$ or both;
6. $\mathfrak{M} \models P \leftrightarrow Q[g]$ iff ($\mathfrak{M} \models P[g]$ iff $\mathfrak{M} \models Q[g]$);
7. $\mathfrak{M} \models \forall x\ P[g]$ iff for every $d \in D^{\mathfrak{M}}$, $\mathfrak{M} \models P[g[x/d]]$;
8. $\mathfrak{M} \models \exists x\ P[g]$ iff for some $d \in D^{\mathfrak{M}}$, $\mathfrak{M} \models P[g[x/d]]$.

Satisfaction, cont'd

Additionally,

- never $\mathfrak{M} \models \bot[g]$;
- always $\mathfrak{M} \models \top[g]$.

A structure $\mathfrak{M}$ satisfies a sentence P,

$$\mathfrak{M} \models P,$$

if $\mathfrak{M} \models P[g_\emptyset]$ for the empty assignment $g_\emptyset$.

Example

$$D^{\mathfrak{M}} = \{a, b, c\}$$

$$\mathfrak{M}(likes) = \{\langle a, a\rangle, \langle a, b\rangle, \langle c, a\rangle\}$$

$$\mathfrak{M} \models \exists x \exists y (Likes(x, y) \land \neg Likes(y, y))$$
$$\mathfrak{M} \models \neg \forall x \exists y (Likes(x, y) \land \neg Likes(y, y))$$

Satisfaction invariance

Proposition Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be structures which have the same domain and assign the same interpretations to the predicates and constant symbols in P. Let g_1 and g_2 be variable assignments that assign the same objects to the free variables in P. Then

$$\mathfrak{M}_1 \models P[g_1] \text{ iff } \mathfrak{M}_2 \models P[g_2]$$

First-order validity and consequence

A sentence P is a first-order consequence of a set $\mathcal{T}$ of sentences if and only if every structure that satisfies all the sentences in $\mathcal{T}$ also satisfies P.

A sentence P is a first-order validity if and only if every structure satisfies P.

A set $\mathcal{T}$ of sentences is called first-order satisfiable, if there is a structure satisfies each sentence in $\mathcal{T}$.

Soundness of $\mathcal{F}$ for FOL

Theorem If $\mathcal{T} \vdash S$, then S is a first-order consequence of $\mathcal{T}$.

Proof: By induction over the derivation, we show that any sentence occurring in a proof is a first-order consequence of the assumption in force in that step.
An assumption is *in force* in a step if it is an assumption of the current subproof or an assumption of a higher-level proof.
For rules involving subproofs that work with fresh constants, we need to use satisfaction invariance.

Completeness of the shape axioms

The basic shape axioms

1. $\neg\exists x(Cube(x) \wedge Tet(x))$
2. $\neg\exists x(Tet(x) \wedge Dodec(x))$
3. $\neg\exists x(Dodec(x) \wedge Cube(x))$
4. $\forall x(Tet(x) \vee Dodec(x) \vee Cube(x))$

SameShape introduction and elimination axioms

1. $\forall x \forall y((Cube(x) \wedge Cube(y)) \rightarrow SameShape(x, y))$
2. $\forall x \forall y((Dodec(x) \wedge Dodec(y)) \rightarrow SameShape(x, y))$
3. $\forall x \forall y((Tet(x) \wedge Tet(y)) \rightarrow SameShape(x, y))$

4. $\forall x \forall y((SameShape(x, y) \wedge Cube(x)) \rightarrow Cube(y))$
5. $\forall x \forall y((SameShape(x, y) \wedge Dodec(x)) \rightarrow Dodec(y))$
6. $\forall x \forall y((SameShape(x, y) \wedge Tet(x)) \rightarrow Tet(y))$

Completeness of the shape axioms

Two structures are isomorphic if there is a bijection between their domains that is compatible with extensions of predicate and interpretation of constants.

Assume the language *Cube*, *Tet*, *Dodec*, *SameShape*

Lemma For any structure satisfying the shape axioms, there is an isomorphic Tarski's world structure.

Theorem Let S be a sentence.

If S is a Tarski's world logical consequence of $\mathcal{T}$, then
S is a first-order consequence of $\mathcal{T}$ plus the shape axioms.